# A Basic Mathematical Analysis of the Ramsey-Cass-Koopmans Model

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# 1 Economic Introduction to the Model

The Ramsey-Cass-Koopmans Model seeks to model consumer consumption and savings on a macroeconomic level. Unlike the Solow Model, the Ramsey-Cass-Koopmans Model allows households to make microeconomic choices about optimal consumption / savings, leading to a variable savings rate.

# 2 Mathematically Deriving the Model

## 2.1 Firms

The model incorporates the behaviors of firms and households. Firms have the production function

$$Y(t) = F(K(t), A(t)L(t)) \quad \text{subject to} \quad Y(t) \geq C(t) + I(t) \quad ^1$$

The law of motion for capital can be presented as follows, assuming a depreciation rate $\delta$:

$$\dot{K}(t) = I(t) - \delta K(t) \quad ^2$$

We can now begin to manipulate our equations to derive an ODE:

$$F(K(t), A(t)L(t)) \geq C(t) + I(t)$$

$$\frac{F(K(t), A(t)L(t))}{A(t)L(t)} \geq \frac{C(t)}{A(t)L(t)} + \frac{I(t)}{A(t)L(t)}$$

Let $c(t) = \frac{C(t)}{A(t)L(t)}, i(t) = \frac{I(t)}{A(t)L(t)}, k(t) = \frac{K(t)}{A(t)L(t)}$:

$$F\left(\frac{K(t)}{A(t)L(t)}, \frac{A(t)L(t)}{A(t)L(t)}\right) \geq c(t) + i(t) \implies F(k(t), 1) \geq c(t) + i(t)$$

Let $f(k(t)) = F(k(t), 1)$:

$$f(k(t)) \geq c(t) + i(t)$$

If we stipulate that productivity grows at rate $n$, $A$ grows at rate $g$:

$$\frac{\dot{L}}{L} = n \quad \text{is separable, yielding ODE solution} \quad L = L_0 e^{nt}$$

$$\frac{\dot{A}}{A} = g \quad \text{is separable, yielding ODE solution} \quad A = A_0 e^{gt}$$

Moreover, we know that the definition of $\dot{k}(t)$ can be written as

$$\dot{k}(t) = \frac{\partial \frac{K(t)}{A(t)L(t)}}{\partial t} \implies \dot{k}(t) = \frac{\dot{K}AL - \left[\dot{A}L + A\dot{L}\right] K}{A^2 L^2}$$

[1] $C(t)$ is aggregate consumption, $I(t)$ is aggregate investment, $K(t)$ is capital, $L(t)$ is labor.
[2] This can also be modeled by the recurrence relation $k_t = k_{t-1}(1-\delta) + i_t$.

$$\implies \dot{k}(t) = \frac{K}{AL}\left[\frac{\dot{K}}{K} - \frac{\dot{A}L}{AL} - \frac{A\dot{L}}{AL}\right] \implies \dot{k}(t) = k(t)\left[\frac{\dot{K}}{K} - g - n\right]$$

We now have that

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}}{K} - g - n$$

By taking our original budget constraint and substituting $I(t) = \dot{K}(t) + \delta K(t)$:

$$F(K(t), A(t)L(t)) \geq C(t) + \dot{K}(t) + \delta K(t) \quad [3]$$

$$F(K(t), A(t)L(t)) = C(t) + \dot{K}(t) + \delta K(t) \implies \dot{K}(t) = F(K(t), A(t)L(t)) - C(t) - \delta K(t)$$

Multiplying both sides of the equation by $\frac{AL}{AL}$:

$$\dot{K}(t) = ALf(k(t)) - ALc(t) - AL\delta k(t)$$

Dividing both sides of the equation by $K$:

$$\frac{\dot{K}(t)}{K(t)} = \frac{AL}{K}\left[f(k(t)) - c(t) - \delta k(t)\right]$$

Plugging in our equation to our derived equation for $\frac{\dot{k}(t)}{k(t)}$ from above:

$$\frac{\dot{k}(t)}{k(t)} = \frac{AL}{K}\left[f(k(t)) - c(t) - \delta k(t)\right] - g - n \quad \text{noting } \frac{AL}{K} = \frac{1}{k(t)}$$

$$\implies \dot{k}(t) = f(k(t)) - c(t) - (\delta + g + n)k(t)$$

## 2.2 Households

Households are utility maximizing, i.e. they want to solve the equation

$$\max_{c} \; U_0 = \int_0^\infty e^{-\rho t} u(C(t))L(t) \, \mathrm{d}t$$

subject to the dynamic budget constraint $\quad \dot{B}(t) = r(t)B(t) + W(t)L(t) - C(t)L(t)$

The model can be presented as follows:

$$\max_{c} \; U_0 = \int_0^\infty e^{-(\rho - n)t} u(c) \, \mathrm{d}t$$

$$\text{subject to} \quad c = f(k) - (\delta + g + n)k - \dot{k}$$

[3] Because firms look to maximize their available resources, they will produce at the edge of their production possibility frontier, allowing us to transform our inequality into an equality.

## 3 Solving as an Optimal Control Problem

We can solve the Ramsey-Cass-Koopmans model by using the current value Hamiltonian. The general form of the current value Hamiltonian (where $\nu(x(t), u(t))$ is the objective function, $u(t)$ is the state variable, $x(t)$ is the control variable, and $\mu(t)$ is the co-state variable) is

$$\mathcal{H}(x(t), u(t)) = \nu(x(t), u(t), t) + \mu(t) f(x(t), u(t))$$

Applied to our specific problem, the state variable is $k(t)$, the control variable is $c(t)$, the co-state variable is $\mu(t)$, the objective function $\nu$ is $U(c(t))$, and the transition function $f(k(t), c(t))$ is $\dot{k}(t) = f(k(t)) - c(t) - (\delta + g + n)k(t)$:

$$\mathcal{H}(k(t), c(t), \mu(t), t) = U(c(t)) + \mu(t) \left[ f(k(t)) - c(t) - (\delta + g + n)k(t) \right]$$

We know that the optimality condition for the Hamiltonian is

$$\frac{\partial \mathcal{H}(k(t), c(t), \mu(t), t)}{\partial c} = 0$$

Taking the derivative with respect to our control variable $c(t)$:

$$U'(c(t)) - \mu(t) = 0$$

We can derive the costate equation by taking the derivative with respect to our state variable $k(t)$:

$$\frac{\partial \mathcal{H}}{\partial k} = (\rho - n)\mu - \dot{\mu}$$

$$u(t) \left[ f'(k(t)) - (\delta + g + n) \right] = (\rho - n)\mu - \dot{\mu}$$

To calculate $\dot{\mu}(t)$, simply take the derivative of $\mu(t)$ with respect to time:

$$\frac{d}{dt} U'(c(t)) = \frac{d}{dt} \mu(t) \implies U''(c(t))\dot{c} = \dot{\mu}(t)$$

Substituting our newfound expressions back into $\frac{\partial \mathcal{H}}{\partial k} = (\rho - n)\mu - \dot{\mu}$:

$$U'(c(t)) \left[ f'(k(t)) - (\delta + g + n) \right] = (\rho - n)U'(c(t)) - U''(c(t))\dot{c}$$

Dividing both sides by $U'(c(t))$:

$$f'(k(t)) - (\delta + g + n) = (\rho - n) - \frac{U''(c(t))}{U'(c(t))}\dot{c}$$

$$\implies \dot{c} = -\frac{U'(c(t))}{U''(c(t))} \left[ f'(k(t)) - (\delta + g + \rho) \right] \quad [4]$$

Now, we can define a specific equation for our utility function for consumption.

$$\text{Let} \quad U(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \quad \sigma \in (0, 1)$$

[4] Note that the equation was rearranged by subtracting $\rho$ and $n$ on both sides.

$$\text{then} \quad U'(c) = (1-\sigma)\frac{c^{1-\sigma-1}}{1-\sigma} = c^{-\sigma} \quad \text{and} \quad U''(c) = -\sigma c^{-\sigma-1}$$

Finally, we can substitute our expressions back into our equation to find

$$\dot{c} = -\frac{c^{-\sigma}}{-\sigma c^{-\sigma-1}} \left[f'(k(t)) - (\delta + \sigma g + \rho)\right]$$

$$\dot{c} = \frac{1}{\sigma} \left[f'(k(t)) - (\delta + \sigma g + \rho)\right]$$ [5][6]

[5]This final derivation gives us the Keynes-Ramsey rule for the rate of change of consumption.

[6]The specific form of this equation is verified in Reference 2.

## 4 Analyzing the Stability of the Model

We now have a model given by two ordinary differential equations:

$$\dot{c}(t) = \frac{1}{\sigma}\left[f'(k(t)) - (\delta + \sigma g + \rho)\right]$$
$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + g + n)k(t)$$

Since we want to find the steady state and analyze the model behavior asymptotically, we can set $\dot{c}(t) = \dot{k}(t) = 0$ to solve for $(k(t) = k^*, c(t) = c^*)$:

$$0 = \frac{1}{\sigma}\left[f'(k^*) - (\delta + \sigma g + \rho)\right]$$
$$\implies f'(k^*) = \delta + \sigma g + \rho$$
$$0 = f(k^*) - c^* - (\delta + g + n)k^*$$
$$\implies c^* = f(k^*) - (\delta + g + n)k^*$$

### 4.1 Analyzing the equilibrium through the assumptions of the model, specifically the No Ponzi Game condition

The No Ponzi Game condition, from an economic perspective, states that debt must grow at a rate less than the real interest rate $r(t)$. Mathematically, it is expressed as

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0$$

where $r(t) - n$ is the growth-corrected real interest rate. In the case of the Ramsey-Cass-Koopmans model, the stipulations of the model are only satisfied with the strict equality

$$lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0 \quad ^7$$

Moreover, we know that the real interest rate $r(t)$ is a function of the rate of change of capital minus depreciation, i.e.

$$r(t) = f'(k^*) - \delta = \delta + \sigma g + \rho - \delta = \sigma g + \rho$$

We can now verify that our equilibrium values $(k^*, c^*)$ satisfy the No Ponzi Game condition:

$$lim_{t\to\infty} a_t e^{-\int_0^t \sigma g + \rho - g - n} ds = 0$$
$$lim_{t\to\infty} a_t e^{-(\sigma g + \rho - g - n)t} ds = 0$$

We can see that since we want the $\lim t \to \infty = 0$, the condition $\sigma g + \rho - g - n > 0$ must hold. In other words, we must have that

$$\rho - n - (1 - \sigma)g > 0$$

This condition was already assumed in our earlier analysis, however, so we know that it must hold here as well, thus proving that the No Ponzi Game condition holds and the $lim_{t\to\infty} a_t e^{-(\sigma g + \rho - g - n)t} ds = 0$.

[7] This is because as $t \to \infty$ the shadow value of per head financial wealth tends to zero in the model, a notion elaborated on much more thoroughly in Reference 2.

## 4.2 Analyzing qualitative behavior via linearization

We can take our system of ordinary differential equations and express them as follows:

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma}\left[f'(k(t)) - (\delta + g + \rho)\right] \\ f(k(t)) - c(t) - (\delta + \sigma g + n)k(t) \end{pmatrix}$$

Since we know that our equations are non-linear, we can linearize them with the Taylor polynomial

$$f(k(t)) = f(k^*) + a(k - k^*)$$

Letting $f(k(t)) = \dot{k}(t)$, we know that at the equilibrium point $k^*$ the derivative $\dot{k} = 0$, so we now have the equation

$$\dot{k} = a(k - k^*)$$

By the same logic, we can linearize $\dot{c}$ to get the equation

$$\dot{c} = b(c - c^*)$$

Rewriting our system of equations, we now have that

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \mathbf{J}(k^*, c^*) \begin{pmatrix} (k - k^*) \\ (c - c^*) \end{pmatrix}$$

We know that

$$\mathbf{J}(k^*, c^*) = \begin{pmatrix} \frac{\partial \dot{k}^*}{\partial k} & \frac{\partial \dot{k}^*}{\partial c} \\ \frac{\partial \dot{c}^*}{\partial k} & \frac{\partial \dot{c}^*}{\partial c} \end{pmatrix}$$

Recall that the functions that yield $k^*, c^*$ are:

$$\dot{c}^* = \frac{1}{\sigma}\left[f'(k^*) - (\delta + \sigma g + \rho)\right]$$

$$\dot{k}^* = f(k^*) - c^* - (\delta + g + n)k^*$$

Thus, we have that the Jacobian matrix

$$\mathbf{J}(k^*, c^*) = \begin{pmatrix} f'(k^*) - \delta - g - n & -1 \\ \frac{1}{\sigma} f''(k^*)c* & 0 \end{pmatrix}$$

By substituting in the steady-state value $f'(k^*) = \delta + \sigma g + \rho$, we have that

$$\mathbf{J}(k^*, c^*) = \begin{pmatrix} \rho - n - (1 - \sigma)g & -1 \\ \frac{1}{\sigma} f''(k^*)c^* & 0 \end{pmatrix}$$

We can calculate the determinant of the Jacobian to classify the type of equilibrium point we have:

$$det(\mathbf{J}(k^*, c^*)) = \frac{1}{\sigma} f''(k^*)c^*$$

We know that the determinant is less than zero because our function $f'(k^*)$ is concave, so $f''(k^*)$ is negative. Moreover, consumption $c^*$ must be positive because negative consumption is impossible. We can also find the eigenvalues $\lambda_1, \lambda_2$ of the matrix by setting the determinant equal to zero.

$$0 = [\rho - n - (1-\sigma)g - \lambda] * -\lambda + \frac{1}{\sigma} f''(k^*)c^*$$

$$0 = \lambda^2 - (\rho - n - (1-\sigma)g)\lambda + \frac{1}{\sigma} f''(k^*)c^*$$

Although we don't have fixed values for these variables, we previously stated that one critical assumption for the model is that $\rho - n - (1-\sigma)g > 0$. Thus, we know that we can simplify our equation as follows:

$$0 = \lambda^2 - c_1\lambda + c_2 \quad c_1 > 0, c_2 < 0$$

This form of quadratic equation can be evaluated as

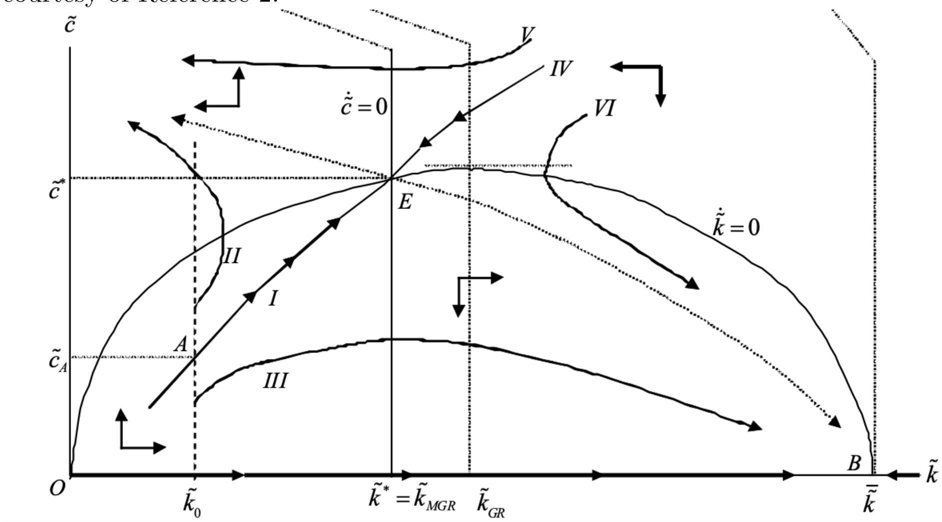
$$\lambda = \frac{c_1 \pm \sqrt{c_1^2 - 4c_2}}{2}$$

We know that since $c_2 < 0$ we can conclude that $c_1^2 - 4c_2 > 0$. If we let

$$c_3 = \frac{\sqrt{c_1^2 - 4c_2}}{2} \quad c_3 > 0 \quad \text{then we have} \quad \frac{c_1}{2} \pm c_3$$

Moreover, since we have that $\frac{c_1}{2} < c_3$, we know that we will have two real eigenvalues, one positive and one negative one. Qualitatively, we know that if we have an equilibrium point with positive and negative real eigenvalues, we have a saddle point. Thus, it's clear that the solution to the Ramsey-Cass-Koopmans model is a saddle point. The solution can be visualized qualitatively as shown in Figure 1 below.

Figure 1: Note that this image was not generated by us and was instead taken courtesy of Reference 2.




Although saddle points are traditionally classified as unstable, a brief economic analysis of the divergent solutions on the unstable arm will make it clear why this solution is stable. For $0 < k_0 < k^*$, it is clear that path $II$ is illogical because $\lim_{k \to 0} c = 0 \neq \infty$. Furthermore, solution path $II$ is also illogical because it violates the existence and uniqueness theorem of ODEs since the path intersects the y-axis at a positive value of $c$. The fact that $k_0 > k^*$ must also tend to the saddle point can be shown via a proof by contradiction. Assume that there exists some $k_1 > k^*$ such that the saddle path does not intersect the region where $k_1 \leq k$. If we let $k_1$ be the smallest value greater than $k$, then the slope $\frac{dc}{dk}$ will grow asymptotically to infinity for $k \to k_1$. We can also observe that $\ln c \to \infty$ as $k \to k_1$. Thus:

$$\frac{d \ln c}{dk} = \frac{\frac{d \ln c}{dt}}{\frac{dk}{dt}} = \frac{\frac{\dot{c}}{c}}{\dot{k}} = \frac{\frac{1}{\sigma}(f'(k) - (\delta + \rho + \sigma g)}{f(k) - c - (\delta + g + n)k}$$

After applying the limits $k \to k_1, c \to infty$, we can see that the denominator grows asymptotically to $-\infty$, which means that $\frac{d \ln c}{dk} \to 0$. This contradicts our earlier statement that $\frac{d \ln c}{dk}$ grows to infinity. Because of our contradiction, it's clear that $k_1 > k^*$ with the properties outlined above cannot exist, so we know that the saddle point must be stable $\forall k_0 \in (0, \infty)$.

## 5 References


1. *A very useful overview of the model.* https://eml.berkeley.edu/~webfac/gourinchas/e202a_f14/Notes_Ramsey_Cass_Koopmans_pog.pdf

2. *A secondary analysis of the model that shows the solution for optimality conditions more clearly using the No Ponzi Game analysis.* https://web.econ.ku.dk/okocg/VV/VV-2010/Lecture%20notes/Ch7-2010-1.pdf

3. *This explanation was referenced extensively for the derivations shown to establish the ODEs in the Ramsey-Cass-Koopmans Model.* https://www.youtube.com/watch?v=GLQQ5nuKy-I

4. *A thread for understanding the geometric depreciation model with $\delta$.* https://economics.stackexchange.com/questions/29632/the-typical-law-of-motion-of-capital

5. *Another general explanation of the model, with particularly helpful diagrams of the equilibrium.* https://en.wikipedia.org/wiki/Ramsey\OT1\textendashCass\OT1\textendashKoopmans_model

6. *Reference matrial for the Jacobian linearization performed in the analysis of the model.* https://www.cds.caltech.edu/~murray/courses/cds101/fa02/caltech/pph02-ch19-23.pdf

7. *An explanation of Euler equations that was tangentially relevant to understanding the model.* https://mitsloan.mit.edu/shared/ods/documents?DocumentID=4171